Koszul duality and manifold calculus

Advances in Homotopy Theory IV

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The rational story

In the 1970's, Quillen and Sullivan pioneered the study of rational homotopy theory, the study of spaces up to equivalence of their rational homotopy groups.

$$\begin{array}{ccc} \mathrm{DGCA}^{f.t.,\geqslant 2}(\mathbb{Q}) & \xrightarrow{\quad K \quad} & \mathrm{DGLA}^{f.t.,\geqslant 1}(\mathbb{Q}) \\ & {\nwarrow}^{\Omega^*_{PL}} \qquad \nearrow & \\ & \mathrm{Top}^{f.t.,\geqslant 2}_{*,\mathbb{Q}} & \end{array}$$

These are equivalences of homotopy theories.

The functor $\bar{\Omega}^*_{PL}$ is a strictly commutative model of reduced singular cochains, and the functor K is Koszul duality between nonunital commutative algebras and Lie algebras.

Example
It is straightforward to check that

$$\bar{\Omega}^*_{PL}(S^{2k}) \simeq \Lambda\langle x, y\rangle$$

where $|x| = 2k, |y| = 4k - 1$ with $dy = x^2$.

$$\bar{\Omega}^*_{PL}(S^{2k-1}) \simeq \Lambda\langle x\rangle$$

where $|x| = 2k - 1$.

The generators and differentials conspire to allow for a map $S^{4k-1}_{\mathbb{Q}} \to S^{2k}_{\mathbb{Q}}$ which can be seen to generate $\pi_{4k-1}(S^{2k}) \otimes \mathbb{Q} \cong \mathbb{Q}$.

Understanding torsion

To what extent does this story extend to integral or $\mathbb{F}_p$ valued cochains?

For finite sets I, J and $a \in I$ let $I \cup_a J = I - \{a\} \sqcup J$.

Definition
An operad O in a symmetric monoidal category $(C, \otimes)$ is a collection of objects $O(I)$ with actions of Σ_I, for all nonempty finite sets I, together with maps for all I, J and $a \in I$

$$O(I) \otimes O(J) \to O(I \cup_a J)$$

which satisfy associativity and equivariance conditions.

Example
The commutative operad com in $(\mathrm{dgVect_k}, \otimes)$ is given by

$$\mathrm{com}(I) = k$$

with all partial composites induced by $k \otimes k = k$.

A cofibrant replacement of com is called an E_∞-operad.

Definition
An algebra A over an operad O in $(C, \otimes)$ is and object $A \in C$ with maps

$$O(I) \otimes A^{\otimes I} \to A$$

satisfying equivariance and associativity conditions.

Proposition
The cup product on $\bar{C}^*(X; k)$ naturally extends to an E_∞-algebra structure.

Theorem (Mandell)
For simply connected spaces X, Y there is a homotopy equivalence $X \simeq Y$, if and only if, there is an equivalence of E_∞-algebras $C^*(X;\mathbb{Z}) \simeq C^*(Y;\mathbb{Z})$.

Theorem (Mandell)
The composite

$$\mathrm{Top}_*^{\geqslant 2} \xrightarrow{\bar{C}^*} \mathrm{Alg}_{E_\infty}(\mathrm{DGVect}_{\mathbb{F}_p}, \otimes) \xrightarrow{K} \mathrm{Alg}_{L_\infty}(\mathrm{DGVect}_{\mathbb{F}_p}, \otimes)$$

takes values in contractible L_∞-algebras.

Higher algebra

Consider the category $(\mathrm{Spec}, \wedge)$.

By setting $\mathrm{com}(I) = S^0$, we can define E_∞-algebras as algebras over a cofibrant replacement of com.

Example
Given a pointed space X, the function spectrum $F(\Sigma^\infty X, S^0)$ obtains an E_∞-algebra structure from the diagonal of X.

This E_∞-structure is a lift of the E_∞-structure on $\bar{C}^*(X)$ to spectra.

Spectral lie algebras

A combination of results of Ching, Francis, Gaitsgory, Harper, and Lurie establish that theories of Koszul duality for algebras and operads in spectra exist, and these lift algebraic Koszul duality in some sense.

With this in mind, one defines the spectral lie operad $\mathrm{lie} := K(\mathrm{com})$, and one obtains a functor

$$\mathrm{Alg}_{E_\infty}(\mathrm{Spec}, \otimes) \xrightarrow{K} \mathrm{Alg}_{L_\infty}(\mathrm{Spec}, \otimes).$$

Unfortunately, the negative results of Mandell still imply that this functor destroys all p-torsion information when applied to $F(\Sigma^\infty X, S^0)$.

Finding the appropriate setting

We implicitly localize at p until stated otherwise. Chromatic homotopy theory studies a certain tower of localization of spec:

$$\begin{array}{ccccccccc}
L_0^f\mathrm{Spec} = \mathrm{Spec}_{\mathbb{Q}} & \longleftarrow & L_1^f\mathrm{Spec} & \longleftarrow & \dots & \longleftarrow & L_n^f\mathrm{Spec} & \longleftarrow & \dots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\mathrm{Spec}_{v_0} = \mathrm{Spec}_{\mathbb{Q}} & & \mathrm{Spec}_{v_1} & & \dots & & \mathrm{Spec}_{v_n} & &
\end{array}$$

The layers of this tower can be "destabilized" by work of Bousfield and Heuts to produce analogs of rational homotopy theory that detect p-torsion.

$$\mathrm{Top}_{v_0} = \mathrm{Top}_{\mathbb{Q}} \qquad \mathrm{Top}_{v_1} \qquad \dots \qquad \mathrm{Top}_{v_n} \qquad \dots$$

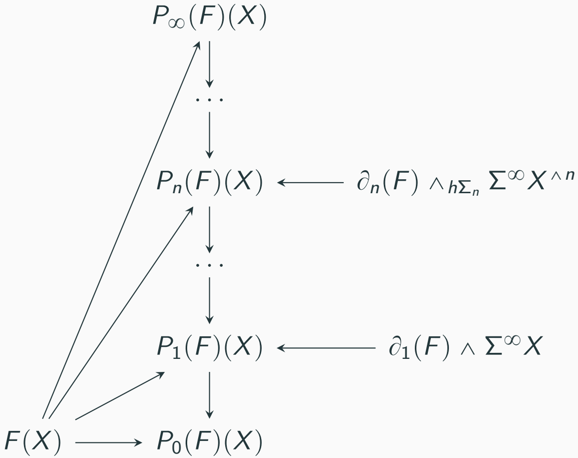
Higher algebra models

Theorem (Heuts)
If $n > 0$*, there is an equivalence* $\mathrm{Alg}_{\mathrm{lie}}(\mathrm{Spec}_{v_n}) \simeq \mathrm{Top}_{v_n}$*.*

Theorem (Behrens-Rezk, Heuts)
If the Goodwillie tower for the identity of $X \in \mathrm{Top}_{v_n}$ *converges, the Koszul dual of* $F(X, S^0_{v_n})$ *is the* v_n *lie algebra model of* X*. In particular, the Goodwillie tower converges for the spheres* $S^d_{v_n}$*.*

Goodwillie calculus

We now drop all localizations of spaces and spectra. Consider a functor $F : \mathrm{Top}_* \to \mathrm{Spec}$. We have Goodwillie approximations



Definition
A right module R over an operad O is a collection of objects $R(I)$ with actions of Σ_I, for all nonempty finite sets I, together with maps for all I, J and $a \in I$

$$R(I) \otimes O(J) \to R(I \cup_a J)$$

which satisfy associativity and equivariance conditions.

Proposition (Arone-Ching)
There is a canonical right lie *module structure on* $\partial_* F$*.*

Theorem (Arone-Ching)
Let $F : \mathrm{Top}_* \to \mathrm{Spec}$*. If* $\partial_* F$ *is levelwise a finite, free* Σ_n*-spectrum, then*

$$P_i(F)(X) \simeq \mathrm{Map}^h_{\mathrm{lie}}((\partial_* \Sigma^\infty \mathrm{Map}(X, -))^{\leqslant i}, (\partial_* F)^{\leqslant i}).$$

In fact, we can compute the derivatives of $\Sigma^\infty \mathrm{Map}_*(X, -)$ explicitly. Define the com module

$$X^\wedge(I) = X^{\wedge I}$$

There is an equivalence of lie modules, $K(\Sigma^\infty X^\wedge) \simeq \partial_*(\Sigma^\infty \mathrm{Map}(X, -))$.

So the Koszul duality of com and lie appears when we differentiate the Yoneda embedding

$$X \to \Sigma^\infty \mathrm{Map}_*(X, -).$$

in order to form the "fake Goodwillie tower" of Arone-Ching.

The little disks operad

Another classic example of an operad is the little n-disks operad E_n.

Definition
The operad E_n has $E_n(I)$ equal to the configuration space of n-disks, labeled by I, in an n-disk. Partial composites are determined by inserting configurations of disks into one another.

Examples of algebras over the E_n operad are n-fold loop spaces. If we apply Σ^∞_+ we obtain a family of operads of spectra whose algebras define increasing levels of commutativity for ring spectra.

(framed) Manifold calculus

In the setting of (framed) manifolds, Weiss developed a calculus for presheaves $F : (\mathcal{M}\mathrm{fld}_n^{\mathrm{fr}})^{\mathrm{op}} \to C$.

$$\begin{array}{ccc} & & T_\infty(F)(M) \\ & \nearrow & \downarrow \\ & & \cdots \\ & & \downarrow \\ & & T_1(F)(M) \\ & \nearrow & \downarrow \\ F(M) & \longrightarrow & T_0(F)(M) \end{array}$$

By restricting F to the subcategory of disjoint unions of disks, F determines a right E_n module. Using some nonstandard notation, we call this right E_n module $\partial_*(F)$.

Proposition (Boavida de Brito-Weiss)
At a framed n-manifold M

$$T_i(F)(M) \simeq \mathrm{Map}_{E_n}((\partial_* \mathrm{Emb}^{\mathrm{fr}}(-, M))^{\leqslant i}, (\partial_* F)^{\leqslant i}).$$

Let E_M denote $\partial_* \mathrm{Emb}^{\mathrm{fr}}(-, M)$. This can be identified with the collection of configuration spaces of disks in M.

By "collapsing disks", one can show that the com module $(M_+)^\wedge$ that we saw earlier is the induction of $(E_M)_+$ along $(E_n)_+ \to \mathrm{com}$.

Comparison of manifold and Goodwillie calculus

By taking inductions and applying Koszul duality, one obtains a map of towers

$$\begin{array}{ccc}
T_\infty(\Sigma^\infty_+\mathrm{Emb}(-, M))(N) & \longrightarrow & P_\infty(\Sigma^\infty\mathrm{Map}(N_+, -)))(M_+) \\
\downarrow & & \downarrow \\
\cdots & & \cdots \\
\downarrow & & \downarrow \\
T_n(\Sigma^\infty_+\mathrm{Emb}(-, M))(N) & \longrightarrow & P_n(\Sigma^\infty\mathrm{Map}(N_+, -)))(M_+) \\
\downarrow & & \downarrow \\
\cdots & & \cdots \\
\downarrow & & \downarrow \\
T_1(\Sigma^\infty_+\mathrm{Emb}(-, M))(N) & \longrightarrow & P_1(\Sigma^\infty\mathrm{Map}(N_+, -))(M_+)
\end{array}$$

Koszul self duality

Theorem (Arone-Ching)
There is a commutative diagram of operads

$$\begin{array}{ccccc} s_n\mathrm{lie} & \longrightarrow & \Sigma^\infty_+ E_n & \longrightarrow & \mathrm{com} \\ \big\downarrow \simeq & & \big\downarrow \simeq & & \big\downarrow \simeq \\ s_n K(\mathrm{com}) & \longrightarrow & s_n K(\Sigma^\infty_+ E_n) & \longrightarrow & K(\mathrm{lie}) \end{array}$$

Conjecture (Ching)
There is an equivalence

$$\mathrm{res}_{\mathrm{lie}} s_{(-n,-n)} \Sigma^\infty_+ E_M \simeq \partial_*(\Sigma^\infty \mathrm{Map}_*(M^+, -)).$$

Koszul dual comparison

By instead restricting from E_n to $s_n\mathrm{lie}$, subject to the conjecture, we obtain a **contravariant** comparison

$$\begin{array}{ccc}
T_\infty(\Sigma^\infty_+\mathrm{Emb}(-,M))(N) & \longrightarrow & P_\infty(\Sigma^\infty\mathrm{Map}(M^+,-)))(N^+) \\
\downarrow & & \downarrow \\
\cdots & & \cdots \\
\downarrow & & \downarrow \\
T_n(\Sigma^\infty_+\mathrm{Emb}(-,M))(N) & \longrightarrow & P_n(\Sigma^\infty\mathrm{Map}(M^+,-)))(N^+) \\
\downarrow & & \downarrow \\
\cdots & & \cdots \\
\downarrow & & \downarrow \\
T_1(\Sigma^\infty_+\mathrm{Emb}(-,M))(N) & \longrightarrow & P_1(\Sigma^\infty\mathrm{Map}(M^+,-))(N^+)
\end{array}$$

Main Results

Theorem (M.)
For a framed n-manifold M, there is an equivalence

$$\mathrm{res}_{\mathrm{lie}} s_{(-n,-n)} \Sigma^\infty_+ E_M \simeq \partial_*(\Sigma^\infty \mathrm{Map}_*(M^+, -)).$$

In the process, we prove a stronger result extending the Koszul self duality of E_n. Let E_{M^+} denote the module of configurations of disks in M^+.

Theorem (M.)
There is a zigzag of equivalences of operads

$$\Sigma^\infty_+ E_n \simeq \cdots \simeq s_n K(\Sigma^\infty_+ E_n)$$

and a compatible zigzag of equivalences of modules

$$\Sigma^\infty_+ E_M \simeq \cdots \simeq s_{(n,n)} K(\Sigma^\infty E_{M^+}).$$

These equivalences are natural with respect to framed embeddings.

Normal bundles of operads

A parametrized spectrum is a space X and a family of spectra indexed by the points of X. This category has a smash product $\bar{\wedge}$ by taking products of the base spaces and smash products of the fibers. Call this category $(\mathrm{ParSp}, \bar{\wedge})$.

Proposition (M.)
There is a factorization

$$\begin{array}{ccc} & & \mathrm{Operad}(\mathrm{ParSp}, \bar{\wedge})) \\ & \overset{\xi_{(-)}}{\nearrow} & \Big\downarrow \mathrm{Th}(-) \\ \mathrm{Operad}(\mathrm{Top}, \times) & \xrightarrow{K(\Sigma^{\infty}_{+}-)} & \mathrm{Operad}(\mathrm{Sp}, \wedge) \end{array}$$

Here, $\mathrm{Th}(-)$ *denotes the Thom complex of a parametrized spectrum, i.e. the spectrum obtained by collapsing the zero section.*

Local Koszul self duality

There is a similar construction of normal bundles $\xi^c_{E_M}$. Taking Thom complexes, we have

$$\mathrm{Th}(\xi^c_{E_M}) \simeq K(\Sigma^\infty E_{M^+}).$$

This assignment of the normal bundle can be seen to be covariant with respect to maps $E_N \to E_M$ induced by $N \subset M$.

Corollary
If $\xi^c_{\mathbb{R}^n}$ has a trivialization as a module over ξ_{E_n}, then for any $U \subset \mathbb{R}^n$ we may choose a module trivialization of ξ^c_U, natural with respect to inclusion.

Koszul self duality inside $\mathbb{R}^n$

Using the Koszul duality of the operad E_n, it is not difficult to deduce the local case

$$\Sigma^\infty_+ E_{\mathbb{R}^n} \simeq \cdots \simeq s_{(n,n)} K(\Sigma^\infty E_{(\mathbb{R}^n)^+}).$$

Such an equivalence implies that $\xi^c_{E_{\mathbb{R}^n}}$ is trivial.

Corollary
If M is a codimension 0 submanifold of $\mathbb{R}^n$, there is a zigzag of equivalences

$$\Sigma^\infty_+ E_M \simeq \cdots \simeq s_{(n,n)} K(\Sigma^\infty E_{M^+}).$$

These equivalences are natural with respect to inclusion.

Globalizing the result

It turns out that the functors

$$M \to \Sigma^\infty_+ E_M$$

$$M \to s_{(n,n)} K(\Sigma^\infty E_{M^+})$$

form Weiss cosheaves, i.e. are sheaves with respect to multilocal covers. Using ∞-categorical techniques due to Ayala-Francis and Lurie, to show these functors agree on $\mathcal{M}\mathrm{fld}^{\mathrm{fr}}_n$, it suffices to check on $\mathrm{Open}(\mathbb{R}^n)$.

Theorem
For a framed n-manifold M, there is a natural zigzag of equivalence of modules

$$\Sigma^\infty_+ E_M \simeq \cdots \simeq s_{(n,n)} K(\Sigma^\infty E_{M^+}),$$

compatible with the Koszul self duality of E_n.

Resolving Ching's conjecture

Recall there is an induction map $E_{M^+} \to (M^+)^\wedge$. By taking Koszul duals, we achieve a map of lie modules

$$K(\Sigma^\infty (M^+)^\wedge) \to \mathrm{res}_{\mathrm{lie}} K(\Sigma^\infty E_{M^+}).$$

Since restriction is Koszul dual to induction, this map is an equivalence. Appealing to the compactly supported self duality of E_M, we obtain a resolution to Ching's conjecture.

Corollary
There is an equivalence

$$\mathrm{res}_{\mathrm{lie}} s_{(-n,-n)} \Sigma^\infty_+ E_M \simeq \partial_*(\Sigma^\infty \mathrm{Map}_*(M^+, -)).$$

Questions?

Verdier duality

Definition
The Verdier dualizing parametrized spectrum $P(X,A)$ of a pair (X,A) is the parametrized spectrum over X such that

$$P(X,A)|_x := \Gamma^c(\Sigma^\infty_X \mathrm{Path}(x,-))$$

Theorem (Klein)
Suppose $A \to X$ is a cofibration such that A and X each have the homotopy type of a compact CW complex and X is connected, then for any $q \in \mathrm{Sp}_X$ there is a zigzag,

$$\mathrm{Th}(P(X,A) \wedge q) \leftarrow \mathrm{Th}(P(X,A) \wedge F_X(P,q)) \to \Gamma^A(q)$$

where the rightmost map is given on the fiber over $x \in X$ by the composition

$$F^A_X(\Sigma^\infty_X X, \Sigma^\infty_X \mathrm{Path}(x,-)) \wedge F_X(\Sigma^\infty_X \mathrm{Path}(x,-), q) \to F^A_X(\Sigma^\infty_X X, q) = \Gamma^A(q)$$

If q is level-fibrant, then after deriving $\wedge, \mathrm{Th}$ these are equivalences