THE UNREASONABLE EFFECTIVENESS OF RIGHT MODULES IN HOMOTOPY THEORY

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Abstract. Here is a list of theorems which can be proven by contemplating the category of right modules over a spectral operad. We expect some of these results, particularly those about chromatic homotopy theory and 1-semiadditivity, are known to experts.

Hypotheses 0.1. *We consider presentable categories C such that $\mathrm{Sp}(C) = \mathrm{Sp}_E$, a Bousfield localization of spectra. All operads are reduced and level-wise finite spectra. All functors are finitary, i.e. commute with filtered colimits. B denotes the covariant theory of bar-cobar duality, and K denotes its dual contravariant theory. These exist for algebras, modules, comodules, etc.*

Proposition 0.2. *Suppose $\partial_* \mathrm{Id}_C$ is level-wise finite. If $F : C \to \mathrm{Sp}_E$ also has level-wise finite derivatives, then we have an equivalence*

$$\partial_* F \simeq B(\mathrm{Hom}^\Sigma(F, \Sigma^\infty_C), \mathrm{CoEnd}(\Sigma^\infty_C), 1)^\vee =: K(\mathrm{Hom}^\Sigma(F, \Sigma^\infty_C)).$$

Here

$$\mathrm{Hom}^\Sigma(-,-) := \{\mathrm{Nat}((-), (-)^{\wedge n})\}$$

is an enrichment in $\mathrm{SymSeq}(\mathrm{Sp}_E)$ and $\mathrm{CoEnd}(-)$ is the endomorphism object.

For any F there is a right $K(\mathrm{CoEnd}(\Sigma^\infty_C))$-module structure on $\partial_*(F)$ which when $\partial_* F$ is level-wise finite, arises in the above way.

Corollary 0.3. *If $c \in C$ is compact, then there is an equivalence*

$$\partial_* \Sigma^\infty_E C(c, -) \simeq K(\{(\Sigma^\infty_C c)^{\wedge n}\}_{n \in \mathbb{N}}).$$

Example 0.4. If C is a reasonable localization of spaces, then

$$\mathrm{CoEnd}(\Sigma^\infty_C) \simeq \mathrm{com}_E,$$

and the formula for the derivatives of a representable functor recovers the localized formula due to Goodwillie. The associated right lie_E module structure agrees with the localization of Arone-Ching's.

Corollary 0.5. *If O is a level-wise finite reduced operad, then*

$$\mathrm{CoEnd}(\Sigma^\infty_{\mathrm{Alg}_O}) \simeq K(O).$$

Hence $\partial_* F$ has a right O-module structure. If A is a compact object in Alg_O, then [1]

$$\partial_* \Sigma^\infty_E \mathrm{Alg}_O(A, -) \simeq K(\mathrm{forget}^{\mathrm{RMod}}_{\mathrm{Coalg}}(B(A))).$$

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[1] This formula is applying the O-bar construction to the algebra A, and then applying $K(O)$-Koszul duality to the associated right $K(O)$-module.

Corollary 0.6. *If O, P are Morita equivalent* [2] *level-finite, reduced operads, then there is some $X \in \mathrm{Pic}(\mathrm{Sp}_E)$* [3] *such that*

$$O \simeq P \wedge \mathrm{End}(X).$$

Corollary 0.7. *If O is a level-finite reduced operad, then any reasonable theories of bar-cobar duality for O-algebras are equivalent.*

Proposition 0.8 (Folklore)**.** *If $B(R)$ is Σ-finite*[4]*, then there is an equivalence*

$$P_\infty B(R, O, -)(A) \simeq \Omega(B(R), B(O), B(A)).$$

In particular, if $B(O)$ is Σ-finite applying this to O establishes that the unit of bar-cobar duality for algebras is an equivalence when the Goodwillie tower converges.

Corollary 0.9 (Ayala-Francis)**.** *For a framed n-manifold, the Goodwillie approximation of the factorization homology of an E_n-algebra A is*

$$P_\infty(\int_M (-))(A) \simeq \int^{M^+} \Sigma^n B(A).$$

Proposition 0.10 (cf. Heuts)**.** *If M is a coanalytic monad* [5] *on the category Sp_{T_h}* [6] *which is induced by an adjunction with right adjoint* forget*, then M is the monad associated to the operad* End(forget)*.*

Corollary 0.11. *Heuts' Lie algebra model of unstable homotopy theory*

$$\mathrm{Top}_{v_h} \simeq \mathrm{Alg}_{\mathrm{lie}}(\mathrm{Sp}_{T_h})$$

can be implemented by taking a v_h periodic space X to $\Phi(X)$ [7] *with the tautological action of $\mathrm{End}(\Phi) \simeq \mathrm{lie}_{T_h}$. If F is a polynomial, then it may be written as*

$$X \to B(\partial_* F, \mathrm{lie}, \Phi(-)).$$

Theorem 0.12 (Behrens-Rezk, Heuts)**.** *If the Goodwillie tower of the identity converges at $X \in \mathrm{Top}_{v_h}$, then there is an equivalence of Lie algebras*

$$\Phi(X) \simeq \Omega(1, \mathrm{com}_{T_n}, \Sigma^\infty_{T_n} X).$$

Theorem 0.13 (Conjecture of Arone-Ching)**.** *There is a notion of divided power right O-modules such that for $F : \mathrm{Top}_* \to \mathrm{Sp}$, $\partial_* F$ has a divided power right* lie*-module structure. This assignement induces an equivalence*

$$\mathrm{Poly}(\mathrm{Top}_*, \mathrm{Sp}) \simeq \mathrm{RMod}^{dp,<\infty}_{\mathrm{lie}}.$$

A version holds for reasonable localizations of Top_*.

Proposition 0.14. *If Alg_O for $O \in \mathrm{Operad}(\mathrm{Sp}_E)$ has*

$$\mathrm{Poly}(\mathrm{Top}_*, \mathrm{Sp}) \simeq \mathrm{RMod}^{dp,<\infty}_{\mathrm{O}}.$$

then Sp_E is 1-semiadditive.

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[2]Morita equivalent means equivalent categories of algebras.

[3]The Picard group $\mathrm{Pic}(\mathrm{Sp}_E)$ is the group of objects invertible with respect to the smash product.

[4]Level-wise equivariantly a finite Borel Σ_i-spectrum.

[5]In the T_n-local setting this just means it splits like the free-forgetful monad of an operad.

[6]This is the telescopic spectrum which appears in chromatic homotopy theory.

[7]$\Phi : \mathrm{Top}_{v_h} \to \mathrm{Sp}_{T_h}$ is the Bousfield-Kuhn functor, a section of $\Omega^\infty_{v_h}$.

Corollary 0.15. *Suppose C is a reasonable localization of n-connected, pointed spaces such that*

$$C \simeq \mathrm{Alg}_O(\mathrm{Sp}_E),$$

then Sp_E is 1-semiadditive.