Koszul duality in functor calculus

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Recollections of operads

Definition
The category of symmetric sequences of spectra $\mathrm{SymSeq}(\mathrm{Spec})$ is $\mathrm{Fun}(\mathrm{FinSet}^{\cong}, \mathrm{Spec})$.

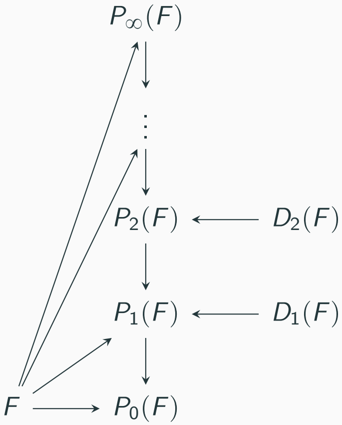
The category $\mathrm{SymSeq}(\mathrm{Spec})$ has a monoidal product $\circ$ given by a certain formula involving smash products, indexed over partitions of finite sets. Heuristically, the spectrum $(X_1 \circ \cdots \circ X_k)(j)$ is the spectrum of rooted, leveled trees with leaves labeled by j with the label of the *ith* level coming from the appropriate X_i.

Definition
The category $\mathrm{Operad}(\mathrm{Spec}, \wedge)$ is $\mathrm{Alg}(\mathrm{SymSeq}(\mathrm{Spec}), \circ)$.

An operad O is reduced if $O(1) = S^0$. All operads will be reduced in this work. There are evident categorical notions of right modules and left modules over an operad. An algebra is a left module concentrated in degree 0.

Functor calculus

Functor calculus is a general method to study functors by associating to them a tower of "polynomial approximations" called the Taylor tower:



Typically, the layers of the tower $D_i(F)$ are easier to understand than then polynomial approximations $P_i(F)$.

Examples of functor calculus

It is often true in the calculus of functors $C \to D$ that the $D_i(F)$ are of a particularly simple form. If $D = \mathrm{Top}_*$, then it is often $\Omega^\infty(X \wedge_{hG_i} S^{\wedge i})$, and if $D = \mathrm{Spec}$, it is often $X \wedge_{hG_i} S^{\wedge i}$ for $x \in \mathrm{Spec}^{BG_i}$

That is, we have a fixed "stabilization" functor $S : C \to \mathrm{Spec}$ and groups G_i which act on it, so that the layers of the tower are classified by G_i-spectra X. This spectrum X is then called the ith derivative $\partial_i F$.

Definition
The symmetric monoidal category $L(C)$ is the opposite of the full symmetric monoidal category generated by $S \in (\mathrm{Fun}(C, \mathrm{Spec}), \wedge)$.

C	S	G_i	$L(C)$
Top_*	Σ^∞	Σ_i	$\Sigma^\infty_+ \mathrm{FinSet}^{\mathrm{sur}}$
Spec	Id	Σ_i	$\Sigma^\infty_+ \mathrm{FinSet}^{\cong}$
Alg_O	TAQ	Σ_i	$\mathrm{Env}(K(O))$
$\mathrm{FinVect}^{\mathrm{inj}}(\mathbb{R})$	$\Sigma^\infty(-)^+$	$O(i)$	$\Sigma^\infty_+ \mathrm{FinVect}^{\mathrm{sur}}(\mathbb{R})$
$\mathrm{FinVect}^{\mathrm{inj}}(\mathbb{C})$	$\Sigma^\infty(-)^+$	$U(i)$	$\Sigma^\infty_+ \mathrm{FinVect}^{\mathrm{sur}}(\mathbb{C})$

Operads and chain rules

Theorem (Arone-Ching)
For functors $F, G : \mathrm{Spec} \to \mathrm{Spec}$ there is an equivalence of symmetric sequences

$$\partial_*(F \circ G) \simeq \partial_* F \circ \partial_* G.$$

Morally, the chain rule implies that the derivatives of a (co)monad in Spec form a (co)operad in Spec and similarly for (co)modules over (co)monads.

Given a pair of functors $F, G : \mathrm{Top}_* \to \mathrm{Top}_*$ there is a comparison

$$FG \to \mathrm{cobar}(F\Omega^\infty, \Sigma^\infty\Omega^\infty, \Sigma^\infty G)$$

Theorem (Arone-Ching)
This differentiates to an equivalence

$$\partial_*(FG) \simeq \mathrm{cobar}(\partial_*(F\Omega^\infty), \partial_*(\Sigma^\infty\Omega^\infty), \partial_*(\Sigma^\infty G)).$$

Additionally, $\partial_*(\Sigma^\infty\Omega^\infty) \simeq \mathrm{com}.$

A multiplicative approach to derivatives

The previous approach relies crucially on the existence of an adjoint to the functor $S : C \to \mathrm{Spec}$. This rules out using comonadic resolutions to study functors

$$\mathrm{FinVect}^{\mathrm{inj}}(\mathbb{R}) \to \mathrm{Spec}.$$

We instead generalize a *different* approach of Arone–Ching which lends itself to proving *product rules*.

Proposition (Arone–Ching)
For $F : \mathrm{Spec} \to \mathrm{Spec}$ there is an equivalence $\mathrm{Nat}(F, \mathrm{Id}^{\wedge n}) \simeq \partial_i(F)^\vee$.

Unfortunately, the analog is not true for functors spaces to spectra:

$$\mathrm{Nat}(F, (\Sigma^\infty)^{\wedge n}) \not\simeq \partial_n(F)^\vee.$$

For instance, $\partial_2(\Sigma^\infty)^\vee \simeq * \not\simeq \mathrm{Nat}(\Sigma^\infty, (\Sigma^\infty)^{\wedge 2}) \simeq S^0$.

Defining away failure

We now fix a functor $F : C \to \mathrm{Spec}$ where C is one of Top_*, $\mathrm{Alg}_{\mathcal{O}}$, Spec, or $\mathrm{FinVect}^{\mathrm{inj}}(k)$.

Definition
The ith Koszul dual derivative $\partial^i(F)$ is $\mathrm{Nat}(F, S^{\wedge i})$.

Example
There is an equivalence

$$\partial^m((\Sigma^\infty)^{\wedge n}) \simeq \bigvee_{f : m \twoheadrightarrow n} S^0 \cong \Sigma^\infty_+ \mathrm{FinSet}^{\mathrm{sur}}(m, n).$$

The Koszul dual derivatives then yield a functor

$$\partial^* : \mathrm{Fun}(C, \mathrm{Spec}) \to \mathrm{Fun}(L(C)^{\mathrm{op}}, \mathrm{Spec}).$$

Koszul duality and right modules

Fix a Spec-enriched symmetric monoidal category W with objects $\mathbb{N}$ and endomorphism objects $\Sigma^\infty_+ G_i$.

Definition
The category of right modules over W, RMod_W, is $\mathrm{Fun}(W^{\mathrm{op}}, \mathrm{Spec})$. It is symmetric monoidal with respect to Day convolution $\otimes$.

Definition
The Koszul dual of W is the opposite of the full symmetric monoidal subcategory of RMod_W generated by $\mathrm{Triv}_W(G_1)$, in the derived sense. The objects are naturally identified with $\mathbb{N}$.

For an operad O, one can use derived indecomposables to show [1]

$$K(\mathrm{Env}(O))(n, 1) \simeq B(1, O, 1)(n)^\vee \simeq K(O)(n).$$

Definition
For $R \in \mathrm{RMod}_W$, the Koszul dual $K(R) \in \mathrm{RMod}_{K(W)}$ is the functor

$$n \to \mathrm{RMod}_W(R, \mathrm{Triv}_W(G_1)^{\otimes n}).$$

[1] The category $\mathrm{Env}(O)$ is defined such that $\mathrm{Alg}_O \cong \mathrm{Fun}^\otimes(\mathrm{Env}(O), \mathrm{Spec})$.

The categories $L(C)$ for $\mathrm{Top}_*, \mathrm{Spec}, \mathrm{FinVect}^{\mathrm{inj}}(k)$

Recall that the category $L(C)$ is the opposite of the category of natural transformations of the smash powers of the functor $S : C \to \mathrm{Spec}$.

Example
The symmetric action on $X^{\wedge n}$ generates

$$L(\mathrm{Spec}) = \Sigma^\infty_+ \mathrm{FinSet}^{\cong}.$$

The symmetric action on $\Sigma^\infty X^{\wedge n}$ together with the diagonal generates

$$L(\mathrm{Top}_*) = \Sigma^\infty_+ \mathrm{FinSet}^{\mathrm{sur}}.$$

Proposition (Arone)
There is an equivalence

$$L(\mathrm{FinVect}^{\mathrm{inj}}(k)) \simeq \Sigma^\infty_+ \mathrm{FinVect}^{\mathrm{inj}}(k)^{\mathrm{op}}.$$

Koszul duality for algebras and the category $L(\mathrm{Alg}_O)$

Definition
Given an O-algebra A, the Koszul dual coalgebra $B(A) \in \mathrm{CoAlg}_{K(O)}$ is given by

$$B(A) := \mathrm{Triv}_O \Sigma_1 \otimes_{\mathrm{Env}(O)} A.$$

The coalgebra structure is induced by functoriality of coends in the first variable. The underlying spectrum is called $\mathrm{TAQ}(A)$.

Recall that for Alg_O, $S = \mathrm{TAQ}$ (Basterra-Mandell).

Proposition
There is an equivalence $L(\mathrm{Alg}_O) \simeq K(\mathrm{Env}(O)) = \mathrm{Env}(K(O))$.

Proposition
Let C be any of $\mathrm{Top}_*, \mathrm{Alg}_O, \mathrm{Spec}, \mathrm{FinVect}(k)^{\mathrm{inj}}$. *There is an equivalence of right modules*

$$\partial^*(X \wedge_{hG_i} S^{\wedge i}) \simeq \mathrm{Free}_{L(C)}(X^\vee \wedge D_{G_i}^\vee).$$

Here D_G is the dualizing spectrum of G. A quick computation then shows $K(\partial^*(X \wedge_{hG_i} S^{\wedge i})) \simeq \mathrm{Triv}_{K((C))}(X)$.

Fake tower

Following Arone-Ching, we define

Definition
The ith fake Taylor approximation is

$$P_i^{\mathrm{fake}}(F)(A) := \mathrm{RMod}_{L(C)}(\partial^* F^{\leqslant i}, \partial^* \Sigma^\infty C(A,-)^{\leqslant i}).$$

By the Yoneda lemma we have natural transformations

$$F \to P_i(F) \to P_i^{\mathrm{fake}}(F).$$

Again by the Yoneda lemma

$$\partial^i(\Sigma^\infty C(A,-)) = S(A)^{\wedge i}.$$

One can compute $D_i^{\mathrm{fake}}(F)(A) := \mathrm{fiber}(P_i^{\mathrm{fake}}(F)(A) \to P_{i-1}^{\mathrm{fake}}(F)(A))$ as

$$K(\partial^* F)(i) \wedge^{hG_i} S(A)^{\wedge i}.$$

Some applications to functor calculus

Theorem
There is an equivalence $\partial_*(F) \simeq K(\partial^*(F))$, and so there is a model for ∂_* which is naturally a right $K(L(C))$-module. Under this equivalence, the map $D_i(F)(A) \to D_i^{\text{fake}}(F)(A)$ is the norm map.

As a consequence, the derivatives of a functor $\mathrm{Alg}_O \to \mathrm{Spec}$ form a right module over $K(K(O)) \simeq O$.

Proposition (Product rule)
As a functor $\partial_* : (\mathrm{Fun}(C, \mathrm{Spec}), \wedge) \to (\mathrm{RMod}_{K(L(C))}, \otimes)$ is symmetric monoidal.

Proof.
Since Koszul duality for right modules is symmetric monoidal, it suffices to prove the result for Koszul dual derivatives. Using the universal property of Day convolution, there is a natural map $\partial^*(F) \otimes \partial^*(G) \to \partial^*(F \wedge G)$ induced by smashing together natural transformations. The question of whether this map is an equivalence can be reduced to homogenous functors. However, the Koszul dual derivatives of homogenous functors are free which implies the result. □

Remarks on bimodule structures

If V is another symmetric monoidal Spec-category, we will define BiMod_{V-W} to be $\mathrm{Fun}^{\otimes}(V, \mathrm{RMod}_W)$. These have a version of Koszul duality landing in $\mathrm{BiMod}_{K(V)-K(W)}$.

Taking inspiration again from Arone–Ching, for a functor $F : C \to \mathrm{Top}_*$, we define

Definition
The unstable Koszul dual derivatives $\partial^* F$ are $\mathrm{Nat}(\Sigma^\infty F, S^{\wedge *})$.

These form a $(\Sigma^\infty_+ \mathrm{FinSet}^{\mathrm{sur}} - L(C))$-bimodule and by repeating the fake Taylor tower construction in the unstable setting, we can produce unstable analogs of many of the aforementioned results.

Applications to Goodwillie calculus

Let $\mathrm{Prim}(-)$ denote the (derived) primitives of a coalgebra.

Proposition (cf. Francis-Gaitsgory, Amabel, Ching-Harper, Behrens-Rezk + Heuts)
Let A be an O-algebra; there is a map $A \to \mathrm{Prim}(B(A))$. If O is Σ-finite, this is an equivalence if the Goodwillie tower of $\mathrm{forget} : \mathrm{Alg}_O \to \mathrm{Spec}$, converges at A.

Proof.
It is easy to show $\partial_* \mathrm{forget} = O$, $\partial^* \mathrm{forget} = \mathrm{Triv}_{K(O)}(\Sigma_1)$, and that $\partial^* \Sigma^\infty \mathrm{Alg}_O(A, -) \simeq B(A)$, where we consider $B(A)$ as a right module using the coalgebra structure maps.

$$\begin{aligned} A &\xrightarrow{\simeq} P_\infty(\mathrm{forget})(A) \\ &\xrightarrow{\simeq} P_\infty^{\mathrm{fake}}(\mathrm{forget})(A) \quad (\Sigma - \mathrm{finiteness}) \\ &\simeq \mathrm{RMod}_{K(O)}(\mathrm{Triv}_{K(O)}(\Sigma_1), B(A)) \\ &:= \mathrm{Prim}(B(A)) \end{aligned}$$

□

Koszul self duality

Theorem (cf. Ching–Salvatore)
The little disks operad E_n is Koszul self dual:

$$\Sigma^\infty_+ E_n \simeq s_n K(\Sigma^\infty_+ E_n)$$

For a framed n-manifold M, the right module E_M of disks in M satisfies a similar self duality

$$\Sigma^\infty_+ E_M \simeq s_{(n,n)} K(\Sigma^\infty E_{M^+}).$$

For an E_n-algebra A let $\int_M A$ denote factorization homology; for an E_n-coalgebra C let $\int^M C$ denote factorization cohomology.

Lemma
There are equivalences $\partial_* \int_M \simeq \Sigma^\infty_+ E_M$ and $\partial^* \int_M \simeq K(\Sigma^\infty_+ E_M)$.

Poincaré/Koszul duality

Theorem (cf. Ayala-Francis)

For a framed n-manifold M, there is a map $\int_M A \to \int^{M^+} \Sigma^n B(A)$ which is an equivalence if the Goodwillie tower of $\int_M A$ converges.

Proof.
Note that $\Sigma^\infty_+ E_M$ is Σ-finite since M is a manifold (assumed to be tame).

$$\begin{aligned}
\int_M A &\xrightarrow{\simeq} P_\infty(\int_M)(A) \\
&\xrightarrow{\simeq} P_\infty^{\mathrm{fake}}(\int_M)(A) \\
&\simeq \mathrm{RMod}_{K(\Sigma^\infty_+ E_n)}(K(\Sigma^\infty_+ E_M), B(A)) \\
&\simeq \mathrm{RMod}_{\Sigma^\infty_+ E_n}(\Sigma^\infty E_{M^+}, \Sigma^n B(A)) \\
&:= \int^{M^+} \Sigma^n B(A)
\end{aligned}$$

□

Relation to the literature

Gregory Arone and Michael Ching. Operads and chain rules for the calculus of functors:

We achieve new constructions of lie operad actions on derivatives in the classical setting of Goodwillie calculus on Top_*. We extend these new constructions to Goodwillie calculus of algebra categories and orthogonal calculus.

David Ayala and John Francis. Poincaré/Koszul duality

We prove/recover Poincaré/Koszul duality for E_n-algebras in spectra (strictly speaking only E_n^{fr}-algebras were dealt with before). This answers some previously open questions on the relation of Koszul self duality of E_n and factorization homology, as well as providing intuition for why Goodwillie calculus has to appear in Poincaré/Koszul duality.

Relation to the literature cont.

Michael Ching. Infinity-operads and Day convolution in Goodwillie calculus:

Our approach is Koszul dual to Ching's more universal approach. Our models of Koszul dual derivatives are often small and geometric. This suggests that combined usage would be very powerful.

Hadrien Espic. Koszul duality for categories with a fixed object set:

Our approach to Koszul duality is very similar to what appears in Espic's thesis, and we expect they agree when they are both defined.

Lurie:

Lurie's treatment of Goodwillie calculus is conjectured to be Koszul dual to Ching's approach which suggests it is closely related to our treatment. The description of Koszul duality we use was originally announced by Lurie at Northwestern in 2023.

Where this work appears

One point compactifications of configuration spaces and the self duality of the little disks operad. arxiv.

Koszul self duality of manifolds. arxiv.

Norm maps and Koszul duality in Goodwillie calculus. in preparation.

Derivatives in orthogonal calculus. joint with Niall Taggart. in preparation.

`https://sites.google.com/view/nialltaggartmath`

End!

Thank you!