A COVARIANT MANIFOLD CALCULUS IN THE STYLE OF GOODWILLIE

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1. Weiss (co)sheaves

These notes are written to accompany a talk at University of Victoria.

Let X be a smooth n-manifold. Fix and ∞-category C which admits (co)limits. Throughout this section, we allow $n = \infty$ and use the convention that an ∞-manifold is just a CW-complex [1].

Definition 1.0.1. A functor $F : \mathrm{Open}(X)^{\mathrm{op}} \to C$ satisfies *descent* for a covering $\{U_i\}_{i\in I}$, if there is an equivalence

$$F(X) \xrightarrow{\simeq} \lim_{S\subset I} F\left(\bigcap_{i\in S} U_i\right),$$

where S ranges over the finite subset of I.

Definition 1.0.2. A k-good Weiss cover of an n-manifold X is an open cover $\{U_i\}_{i\in I}$ for which any U_i has at most k path components each of which is contractible. We require that the intersection of any finite number of opens has at most k-components each of which is contractible. The cover is required to have the property that any set of k or fewer points is contained inside some open U_i.

If $n < \infty$ every smooth manifold has a k-good Weiss cover, even one in which the intersections are diffeomorphic to $\bigsqcup_i \mathbb{R}^n$. If $n = \infty$, every CW-complex is equivalent to one which has a k-good Weiss cover. [2]

Definition 1.0.3. For a functor $F : \mathrm{Open}(X)^{\mathrm{op}} \to C$, the kth embedding calculus approximation $T_k(F) : \mathrm{Open}(X) \to C$ is the sheafification of F with respect to k-good Weiss covers.

Remark 1.0.4. *We are primarily interested in topological presheaves, i.e. those defined on the entire ∞-category Mfld_n. In these cases, one can compute $T_k(F)(M)$ by the right Kan extension (factorization cohomology) of the restriction to the category of manifolds diffeomorphic to a disjoint union of disks:*

$$\int^M F|_{\mathrm{Disk}_n} := \int_{\bigsqcup_i \mathbb{R}^n \in \mathrm{Disk}_n} C(\mathrm{Emb}(\bigsqcup_i \mathbb{R}^n), M), F(\bigsqcup_i \mathbb{R}^n)).$$

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[1] In reality, the stabilization of the ∞-category of (tame) manifolds and embeddings with respect to cartesian product with $\mathbb{R}$ is $\mathrm{Top}^{\mathrm{fin}}_{/BO}$, not quite Top_*.

[2] There are broader classes of k-covers which define the same category of sheaves.

This latter end is just the space of natural transformations $\mathrm{Nat}(\mathrm{Emb}(-, M)|_{\mathrm{Disk}_n}, F|_{\mathrm{Disk}_n})$

These sheafifications exist and are equipped with a map $F \to T_k(F)$ for formal reasons. As a consequence, we have a tower

$$\begin{array}{ccc} & & T_\infty(F)(M) := \lim\limits_{i\to\infty} T_i(F)(M) \\ & \nearrow & \downarrow \\ & & \cdots \\ & & \downarrow \\ & \nearrow & T_1(F)(M) \\ & & \downarrow \\ F(M) & \longrightarrow & T_0(F)(M) \end{array}$$

We call $\mathrm{fiber}(T_k(F)(M) \to T_{k-1}(F)(M))$ the layers of the embedding calculus tower. Such fibers are classified but have proved largely intractable to study though some computations exist in the rational case.

The fundamental reason geometric topologists are interested in embedding calculus is that it accurately describes spaces of embeddings in high enough codimension:

Theorem 1.0.5 (Goodwillie-Klein-Weiss)**.** *If M is a smooth manifold of handle dimension*[3] *m and N is a smooth n-manifold such that $n - m \geq 3$, then* $T_\infty(\mathrm{Emb}(-, N))(M) \simeq \mathrm{Emb}(M, N)$.

If instead we are dealing with covariant functors $F : \mathrm{Open}(X) \to \mathrm{Top}_*$, we could produce a formally dual theory of Weiss cosheaves, which would form a tower mapping *into* the functor F.

Example 1.0.6. *The assignment $U \mapsto \mathrm{Conf}(U, k)$ satisfies codescent for the k-good Weiss covers.*

The theory of Weiss cosheaves shows up in the study of factorization algebras. If one is interested in studying embeddings, unfortunately, it is difficult to use Weiss cosheaves. This is because the analog of Goodwillie-Klein-Weiss for $\mathrm{Emb}(M, -)$ is far from true, essentially because the spaces $\mathrm{Emb}(M, \bigsqcup_i \mathbb{R}^n)$ contain approximately the same information as i increases, especially if M is connected.

2. Goodwillie calculus

Predating embedding calculus is Goodwillie calculus. Goodwillie calculus studies covariant functors $F : \mathrm{Top}_* \to \mathrm{Spec}$ which preserve 0-maps.

[3]The handle dimension is the minimum over all handle presentations of the largest index of a handle. It may be less than the actual dimension if the manifold is noncompact.

Definition 2.0.1. The kth Goodwillie approximation of F

$$F \to P_k(F)$$

is the initial object in the ∞-category of Weiss k-cosheaves under F.

Unlike cosheafification with respect to k-good Weiss covers, there is no obvious reason why $P_k(F)$ should exist since cosheaves are defined by colimits, and so there is not a formal process to construct maps into them. The fact that such universal cosheaves actually do exist is powerful. Once constructed these approximations are much easier to compute with than those of embedding calculus.

Example 2.0.2. *If F is a Weiss k-cosheaf, then $P_k(F) \simeq F$. We call such functors k-excisive. Some examples of k-excisive functors are*

$$X \mapsto \Sigma^\infty X^{\times k},$$
$$X \mapsto \Sigma^\infty X^{\wedge k}.$$

Definition 2.0.3. We call a functor k-reduced if $P_k(F) \simeq *$. We call a functor k-homogeneous if it is k-excisive and $(k-1)$-reduced.

Example 2.0.4. *The functor*

$$X \mapsto \Sigma^\infty X^{\times k}$$

is not k-homogenous if $k > 1$ since admits a nontrivial natural transformation to the $(k-1)$-excisive functor $\Sigma^\infty X^{\times k-1}$ (projection).

Theorem 2.0.5 (Goodwillie). *Any k-homogeneous functor is of the form*

$$X \mapsto (Z \wedge \Sigma^\infty X^{\wedge k})_{\Sigma_k}$$

for Z a spectrum with a Σ_k-action.

Goodwillie calculus provides systematic and powerful tools to study natural transformations between functors. The author knows of no other way to prove the following:

Corollary 2.0.6. *The space of natural transformations from $\Sigma^\infty X^{\wedge i}$ to $\Sigma^\infty X^{\wedge j}$ is contractible if $i > j$.*

Proof. There are equivalences

$$\mathrm{Nat}(\Sigma^\infty X^{\wedge i}, \Sigma^\infty X^{\wedge j}) \xrightarrow{\simeq} \mathrm{Nat}(P_j(\Sigma^\infty X^{\wedge i}), \Sigma^\infty X^{\wedge j}) \simeq \mathrm{Nat}(*, \Sigma^\infty X^{\wedge j}) = *$$

□

Of course, there are natural transformation in the other direction given by diagonal maps. Understanding these “upward” natural transformations between homogeneous functors is the key to understanding some of the recent approaches to Goodwillie calculus using the lie and E_n operads.

Definition 2.0.7. The Σ_k-spectrum $\partial_k F$ is the Σ_k-spectrum classifying the homogeneous functor

$$D_i(F) := \text{fiber}(P_i(F) \to P_{i-1}(F)).$$

Example 2.0.8. *There is an equivalence*

$$D_i(\Sigma^\infty(-^{\times k}))(X) \simeq \Sigma^\infty X^{\wedge k}.$$

Hence $\partial_i(\Sigma^\infty X^{\wedge k}) \simeq \Sigma^\infty_+ \Sigma_k$. *Further, the map*

$$\Sigma^\infty X^{\wedge k} \simeq D_i(\Sigma^\infty(-^{\times k}))(X) \to P_i(\Sigma^\infty(-^{\times k})(X)) \simeq \Sigma^\infty(X^{\times k})$$

witnesses the stable splitting

$$\Sigma^\infty X^{\times k} \simeq \Sigma^\infty X^{\wedge k} \vee \dots.$$

Example 2.0.9. *There is an equivalence*

$$\partial_i(\Sigma^\infty \text{Map}_*(X, -)) \simeq \Sigma^\infty X^{\wedge i}/(\Delta^{\text{fat}})^\vee.$$

Here Δ^{fat} *is the complement of configuration space and* $\vee$ *is Spanier–Whitehead duality.*

In his thesis, Arone gave an embedding calculus type model for the Goodwillie tower of this functor which is sufficient to make this calculuation. Goodwillie also produced a combinatorial computation of this derivative in his original paper.

3. Zero pointed manifolds

In the previous section, we saw the the homogeneous functors were built out of smash products $X^{\wedge k}$ rather than the categorical products $X^{\times k}$, since the latter has natural transformations to excisive functors of degree $< k$. In the context of manifold calculus, the role of $X^{\times k}$ is usually played by the configuration space $\text{Conf}(M, k)$. This suggests the question: what plays the role of $X^{\wedge k}$?

Definition 3.0.1. A zero-pointed n-manifold is a pointed topological space W such that $W - *$ is an n-manifold. A zero-pointed embedding of zero-pointed manifolds W to W' is a pointed map which is a smooth embedding away from $* \in W'$.

Definition 3.0.2. For a zero-pointed manifold W, we let $C(W, k)$ denote

$$\{(x_1, \dots, x_k) | x_i = x_j \implies i = j \text{ or } x_i = x_j = *\} \subset W^{\times k}.$$

We let $\bar{C}(W, i)$ denote the quotient of $C(W, k)$ by the fat wedge: the subspace of $W^{\times k}$ for which any point is the basepoint.

The space $C(W, i)$, which has a natural basepoint $(*, \dots, *)$, is the analog of the product space $X^{\times i}$, which has a natural basepoint $(*, \dots, *)$. The quotient $\bar{C}(W, i)$, with natural basepoint $*$, is the analog of the smash product $X^{\wedge i}$, with natural basepoint $*$.

Our results concern a variant of the category of zero-pointed manifolds which has the property that $n = \infty$ recovers the category of pointed, finite CW-complexes and pointed maps.

Definition 3.0.3. Let $\mathrm{ZMfld}_n^{\mathrm{fr}}$ denote the category of (tame) framed zero-pointed manifolds, i.e. those zero-pointed manifolds equipped with a framing of the complement of the distinguished point. The morphisms are the zero-pointed framed embeddings. [4]

4. Zero pointed framed manifold calculus

Suppose we have a functor $F : \mathrm{ZMfld}_n^{\mathrm{fr}} \to \mathrm{Spec}$ which sends constant zero-pointed embeddings to zero maps of spectra. Since we have analogs of Weiss k-cosheaves, $X^{\wedge k}$, and our category stabilized to $\mathrm{Top}_*^{\mathrm{fin}}$, one might expect a good theory of Goodwillie calculus to exist. Fix $n < \infty$.

Definition 4.0.1. The kth Goodwillie approximation of F

$$F \to P_k(F)$$

is the initial object in the ∞-category of Weiss k-cosheaves under F.

In certain cases, we can show these approximations exist by explicitly constructing them.

Example 4.0.2. *If* $F = \Sigma^\infty \mathrm{ZEmb}^{\mathrm{fr}}(M_+, -)$, *then* $P_i(\Sigma^\infty \mathrm{ZEmb}^{\mathrm{fr}}(M_+, -))(N_+)$ *exists and agrees with the embedding calculus approximation* $T_i(\Sigma_+^\infty \mathrm{Emb}^{\mathrm{fr}}(-, N))(M)$, *computed in the subcategory of* π_0*-surjective embeddings. We will use the superscript* sur *to denote passage to the category of* π_0*-surjective embeddings.*

Taking $n = \infty$, *this recovers the model of the Goodwillie tower of* $\Sigma^\infty \mathrm{Map}_*(X, -)$ *of Arone's thesis.*

In contrast to the Weiss cosheafification of $\Sigma^\infty \mathrm{ZEmb}^{\mathrm{fr}}(M_+, -)$, this is a very nontrivial approximation, and there are situations where it conjecturally converges.

Although our definition of the Goodwillie approximations is identical to the definition for Top_*, there is no reason that this approximation (should it exist) would produce the same result. This is because not all Weiss cosheaves on $\mathrm{ZMfld}_n^{\mathrm{fr}}$ will extend to Top_*.

In fact, these approximations do exist.

Theorem 4.0.3 (M.)**.** *The* k*th Goodwillie approximation of* F *exists. Further, the homogeneous functors are classified by* $(Z \wedge \Sigma^\infty \bar{C}(W, k))_{\Sigma_k}$.

Remark 4.0.4. *A surprising consequence of this theorem is that*

$$M \to \Sigma^\infty M^{\wedge k}$$

admits nontrivial natural transformations to

$$M \to \Sigma^\infty \bar{C}(W, j)$$

for some $j < k$. *Such natural transformations can be written down explicitly using Ching's theory of bar-cobar duality for operads and right modules.*

[4] The definition of a zero-pointed framed embedding is more subtle than the unframed counterpart. It is made more complicated by the necessity to give the set of all zero-pointed framed embeddings the correct topology.

We also demonstrate structural results for derivatives.

Proposition 4.0.5 (M.). *The derivatives of F have the structure of a right module over the shifted (reduced) E_n operad in spectra. In terms of disk categories this means that some suspension of the symmetric sequence $\partial_*(F)$ can be upgraded to a presheaf on the category $\mathrm{Disk}_n^{\mathrm{fr,sur}}$. This completely classifies the Goodwillie tower.*

Finally, we have a Poincaré/Koszul duality theorem which computes the manifold Goodwillie tower in terms of Weiss sheafification see Remark 1.0.4.

Theorem 4.0.6 (M.). *There is an equivalence*

$$P_\infty(F)(M) \xrightarrow{\simeq} \int^{M^+} s_{(n,n)}\partial_*(F)$$

where $\int^{M^+}$ denotes factorization cohomology over M^+ of a presheaf on $\mathrm{Disk}_n^{\mathrm{fr,sur}}$. [5]

If we take F to be factorization homology with coefficients in an E_n algebra A, this ultimately recovers a result of Ayala-Francis with a fundamentally different proof. Ayala-Francis proceed by Goodwillie calculus in the category of E_n-algebras, while we proceed by Goodwillie calculus in the category of framed manifolds. Miraculously both functor calculus towers agree.

[5] If F is a Weiss cosheaf, then $\partial_*(F)$ is Koszul dual to the restriction of F to the disk category.